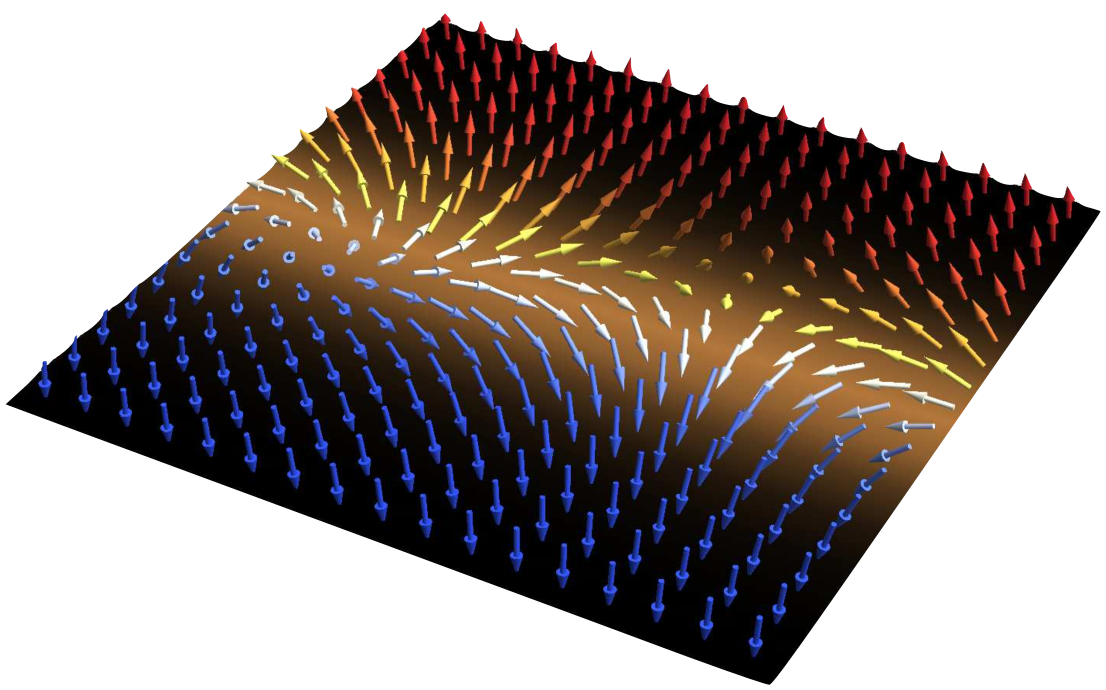
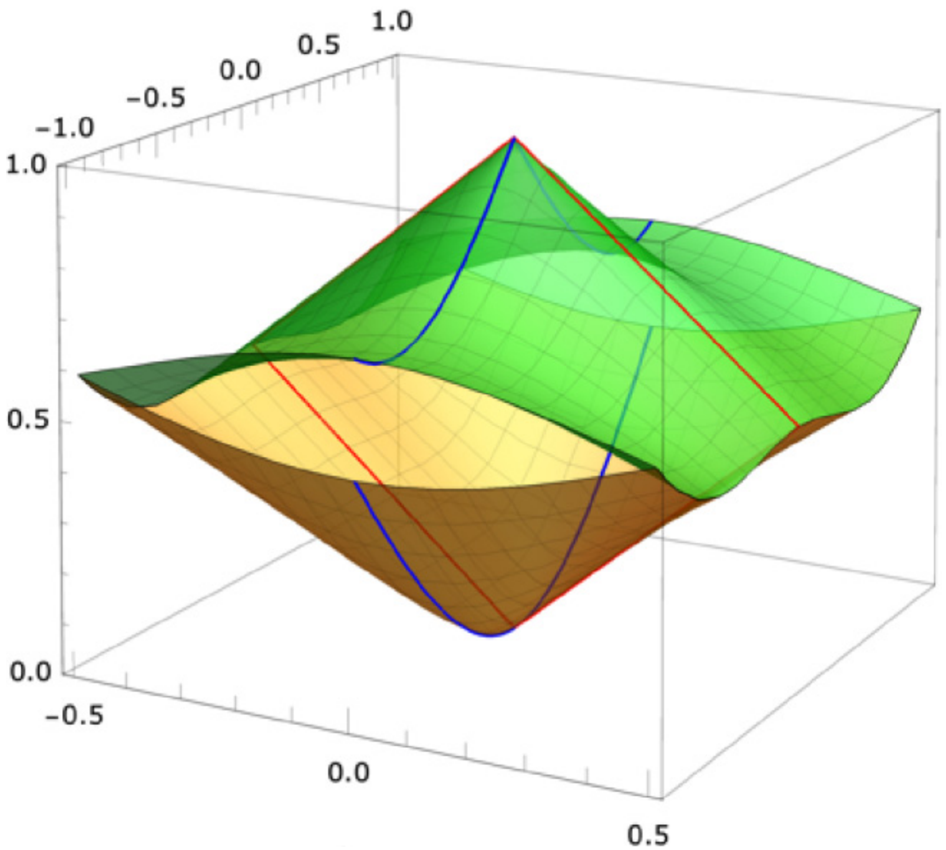
Instanton, inhomogeneous phase, and NG mode in chiral magnets





Masaru Hongo (Univ. of Illinois Chicago)

2021/11/03, Solitons at Work Seminars

MH-Fujimori-Misumi-Nitta-Sakai , PRB 101 (2020) 10, 104417, PRB 104 (2021) 13, 134403

Chiral magnets

Spin systems equipped with Dzyaloshinsky-Moriya interaction

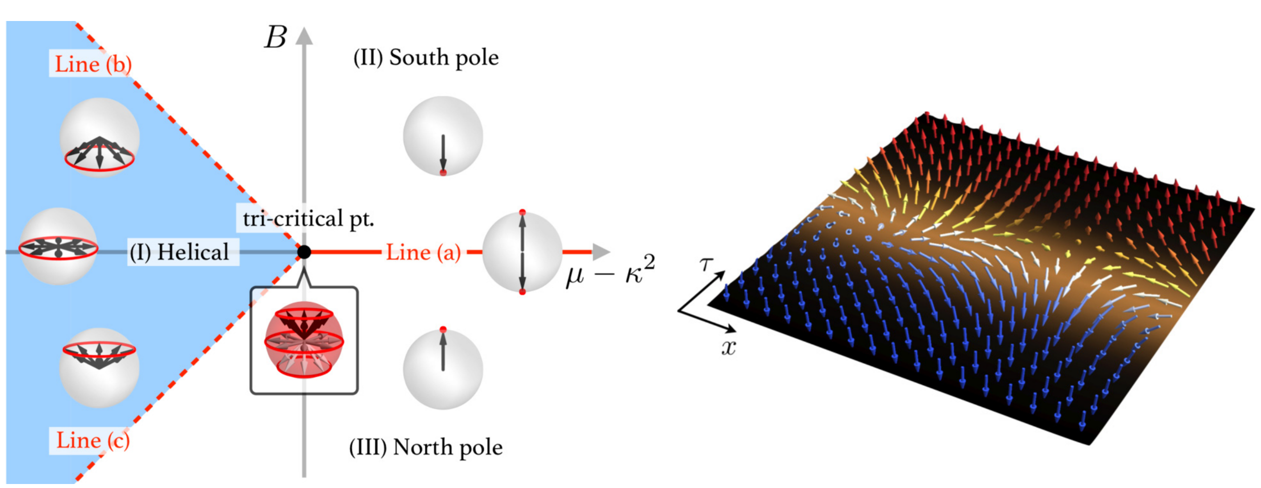
[Dzyaloshinsky (1958), Moriya (1960)]

◆ Many experimental realizations!

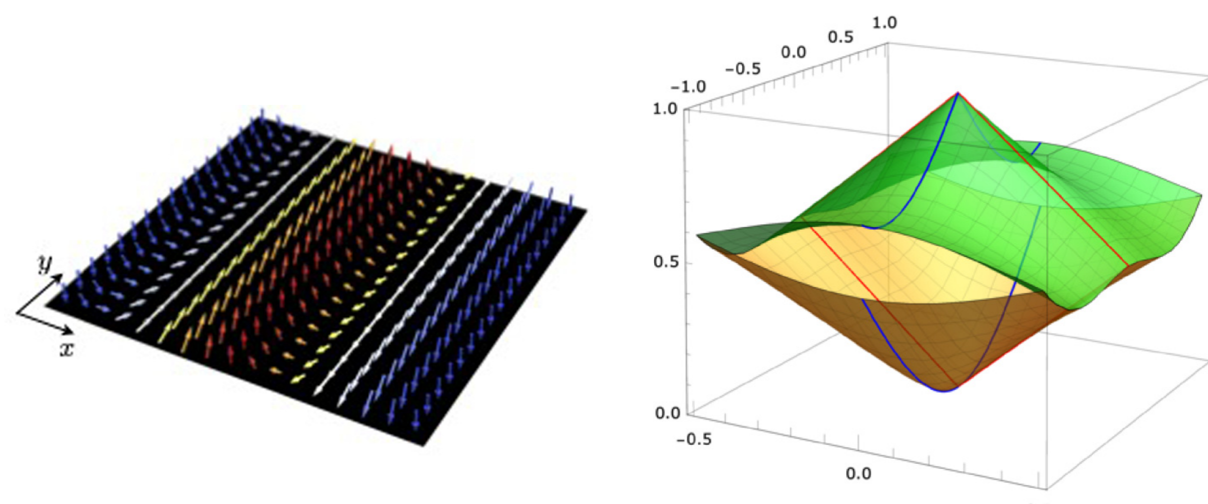
×

◆ <u>Interesting theoretical aspects!!</u>

Instantons in 1+1d antiferro magnet



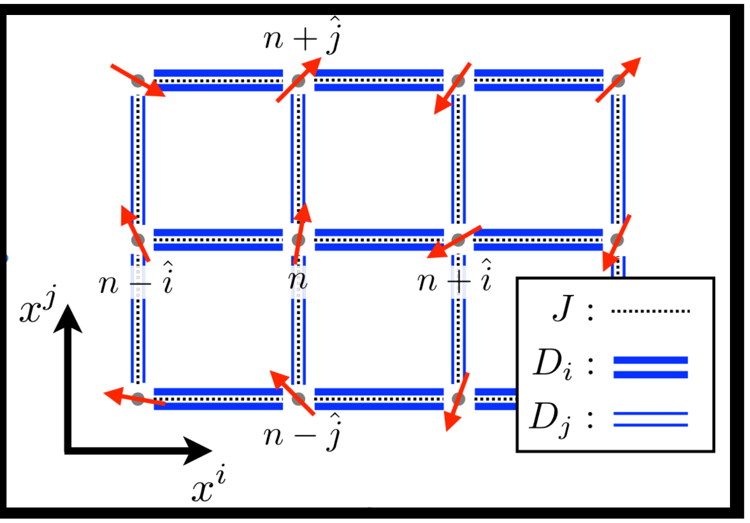
Helical/spiral phases and NG modes



Outline

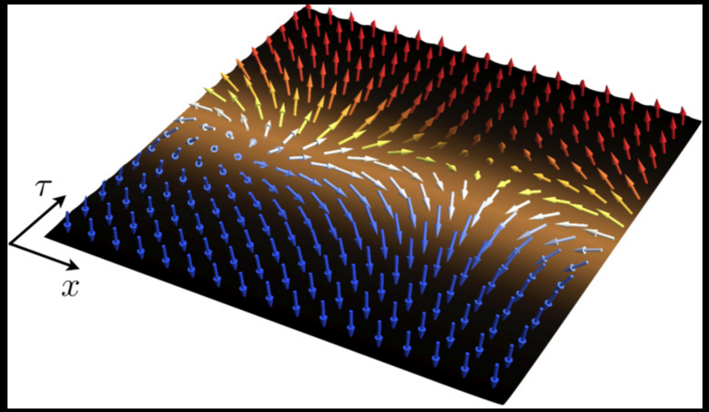
Formulation:

Background field (spurion) method
for O(3) nonlinear sigma model



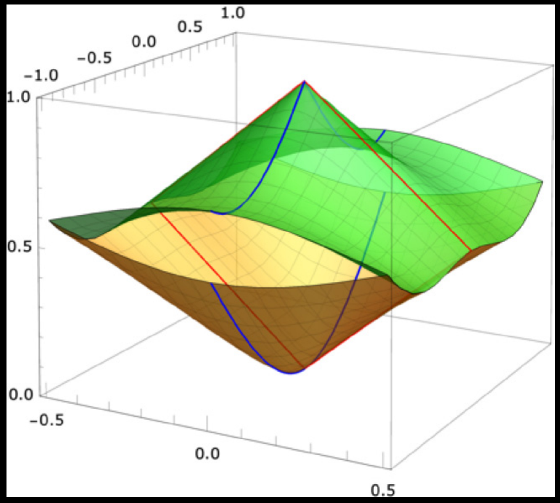
Instantons in 1+1d antiferro magnet :

Various instanton solutions
Equivalence theorem



Helical/spiral phases and NG modes:

Inhomogeneous ground states
Several types of NG modes



What is DM interaction?

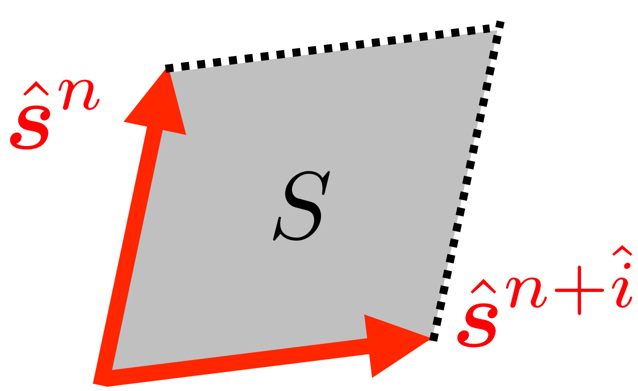
$$\hat{H} = \sum_{n} \sum_{i=1}^{d} \left[\frac{J}{2} \left(\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^{n} \right)^2 + \boldsymbol{D}_i \cdot \left(\hat{\boldsymbol{s}}^{n} \times \hat{\boldsymbol{s}}^{n+\hat{i}} \right) \right] + \sum_{n} (\hat{\boldsymbol{s}}^{n})^t C \hat{\boldsymbol{s}}^{n}$$

Dzyaloshinsky-Moriya interaction

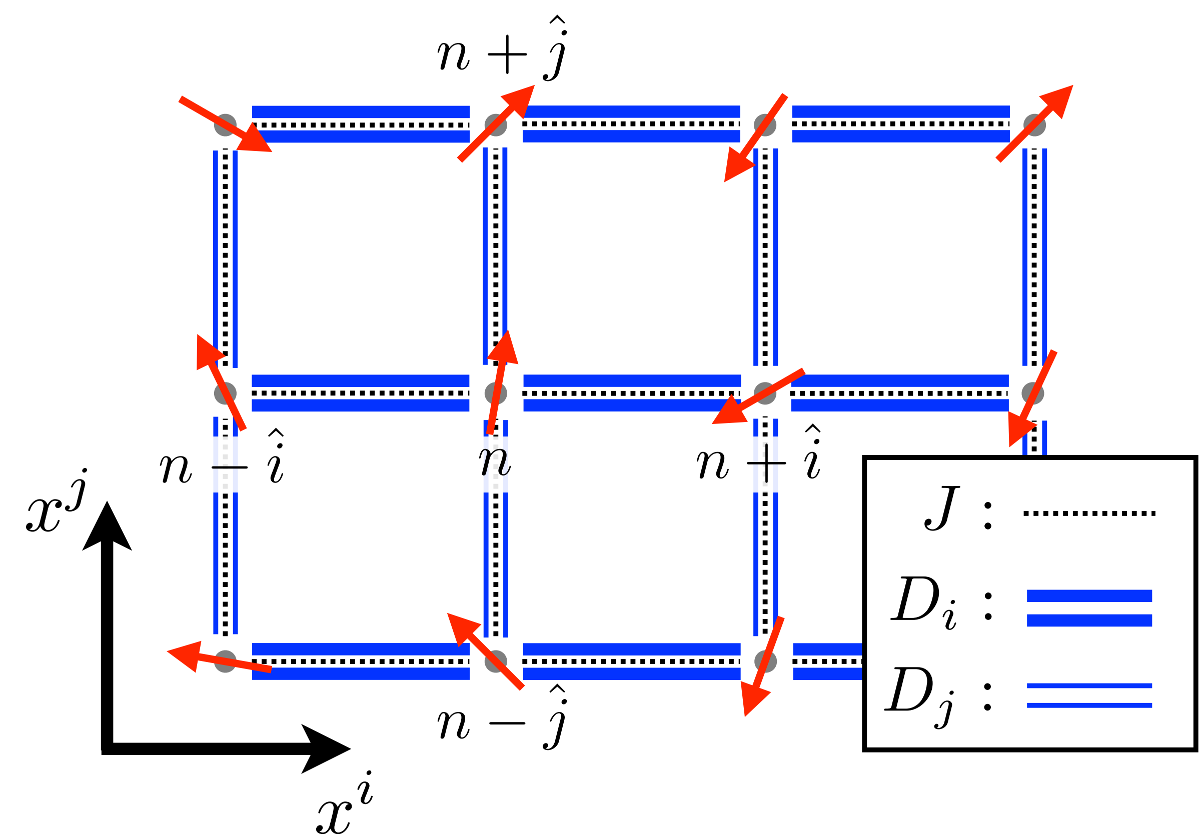
Anisotropic pot. (single-ion aniso.)

◆ Important property

Proportional to **vector product**!



➡ Increase (or decrease) S!



➡ Favor **inhomogeneous** spin configuration!

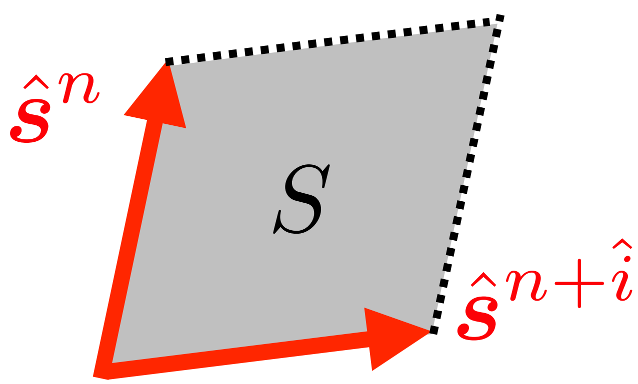
What is DM interaction?

$$\hat{H} = \sum_{n} \sum_{i=1}^{d} \left[\frac{J}{2} \left(\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^{n} \right)^2 + \boldsymbol{D}_i \cdot \left(\hat{\boldsymbol{s}}^{n} \times \hat{\boldsymbol{s}}^{n+\hat{i}} \right) \right] + \sum_{n} (\hat{\boldsymbol{s}}^{n})^t C \hat{\boldsymbol{s}}^{n}$$

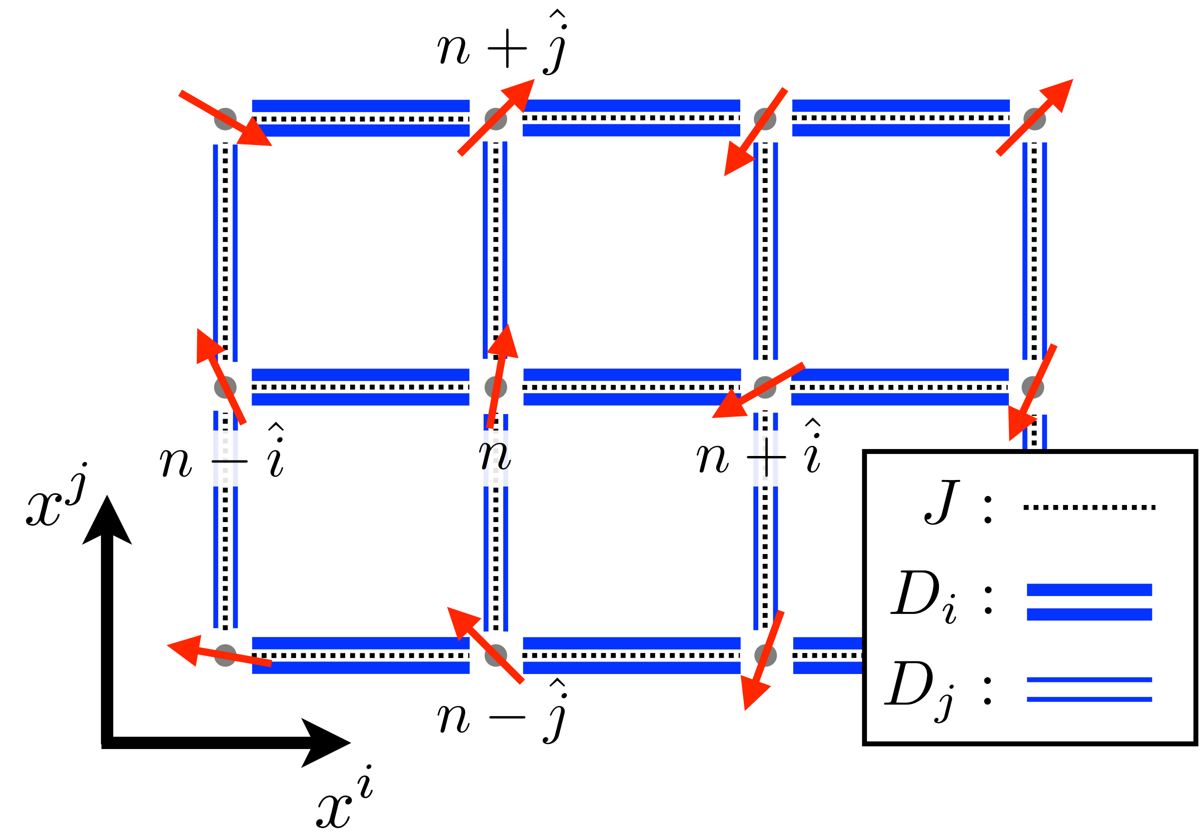
Described by bkg. SO(3) gauge field!!

◆ **Important property**

Proportional to **vector product**!



➡ Increase (or decrease) S!



➡ Favor **inhomogeneous** spin configuration!

DM interaction ≒ SO(3) gauge field

$$\hat{H} = \sum_{n}\sum_{i=1}^{d}\left[\frac{J}{2}(\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^{n})^2 + \boldsymbol{D}_i \cdot (\hat{\boldsymbol{s}}^{n} \times \hat{\boldsymbol{s}}^{n+\hat{i}})\right] + \sum_{n}(\hat{\boldsymbol{s}}^{n})^t C \hat{\boldsymbol{s}}^{n}$$

Introduce SO(3) lattice gauge field (= link variables)

$$\hat{H}'_0 = \frac{J}{2}\sum_{n}\sum_{i=1}^{d}[U(n, n+\hat{i})\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^{n}]^2 \quad \text{with} \quad U(n, n+\hat{i}) = \mathrm{e}^{\mathrm{i}aA_i^a t_a}$$

<u>SO(3) gauge inv.</u> : $[g(n) \in SO(3)]$

$$\begin{cases} \hat{\boldsymbol{s}}_n \to g(n)\hat{\boldsymbol{s}}_n \\ U(n, n+\hat{i}) \to g(n)U(n, n+\hat{i})g(n+\hat{i})^t \end{cases}$$

Collect $O(a^2)$ terms by expanding w.r.t lattice spacing a

$$\hat{H}'_0 = \sum_{n}\sum_{i=1}^{d}\left[\frac{J}{2}(\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^{n})^2 + Ja\boldsymbol{A}_i \cdot (\hat{\boldsymbol{s}}^{n+\hat{i}} \times \hat{\boldsymbol{s}}^{n})\right] + \sum_{n}\frac{Ja^2}{2}(\hat{\boldsymbol{s}}^{n})^t(A_i^a t_a)^2 \hat{s}^{n}$$

DM interaction ≒ SO(3) gauge field

$$\hat{H} = \sum_n \sum_{i=1}^{d} \left[\frac{J}{2} (\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^n)^2 + \boldsymbol{D}_i \cdot (\hat{\boldsymbol{s}}^n \times \hat{\boldsymbol{s}}^{n+\hat{i}}) \right] + \sum_n (\hat{\boldsymbol{s}}^n)^t C \hat{\boldsymbol{s}}^n$$

Introduce SO(3) lattice gauge field (= link variables)

$$\hat{H}_0' = \frac{J}{2} \sum_n \sum_{i=1}^{d} [U(n, n+\hat{i}) \hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^n]^2 \quad \text{with} \quad U(n, n+\hat{i}) = \mathrm{e}^{\mathrm{i} a A_i^a t_a}$$

<u>SO(3) gauge inv.</u> : $\begin{cases} \hat{\boldsymbol{s}}_n \to g(n) \hat{\boldsymbol{s}}_n \\ U(n, n+\hat{i}) \to g(n) U(n, n+\hat{i}) g(n+\hat{i})^t \end{cases}$

$[g(n) \in SO(3)]$

Collect O(a^2) terms by expanding w.r.t lattice spacing *a*

$$\hat{H}_0' = \sum_n \sum_{i=1}^{d} \left[\frac{J}{2} (\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^n)^2 + Ja \boldsymbol{A}_i \cdot (\hat{\boldsymbol{s}}^{n+\hat{i}} \times \hat{\boldsymbol{s}}^n) \right] + \sum_n \frac{Ja^2}{2} (\hat{\boldsymbol{s}}^n)^t (A_i^a t_a)^2 \hat{s}^n$$

$$\Rightarrow \quad A_i^a = (Ja)^{-1} D_i^a, \quad C_{\mathrm{cr}} = \frac{1}{2J} (D_i^a t_a)^2$$

O(3) nonlinear sigma model

$$\hat{H} = \sum_n \sum_{i=1}^{d} \left[\frac{J}{2} (\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^n)^2 + \boldsymbol{D}_i \cdot (\hat{\boldsymbol{s}}^n \times \hat{\boldsymbol{s}}^{n+\hat{i}}) \right] + \sum_n (\hat{\boldsymbol{s}}^n)^t C \hat{\boldsymbol{s}}^n$$

General anisotropic pot. C = field behaving as symmetric tensor rep.

$$W(n) \equiv C - C_{\rm cr} \to g(n) W(n) g(n)^t$$

◆ **A way to describe DM int. & anisotropic pot.**

Write down action with bkg. SO(3) gauge +SO(3) tensor rep. fields, and fix them as

$$A_i^a = (Ja)^{-1} D_i^a, \quad W = C - C_{\rm cr} \quad \text{with} \quad C_{\rm cr} \equiv \frac{1}{2J} (D_i^a t_a)^2$$

In continuum limit, spin = $n^a \ (a = 1,2,3)$ with $(n^a)^2 = 1$

◆ **Most generral effective Lagrangian at O(p²)**

$$\mathcal{L}_{\rm eff} = \frac{m(n^2 \partial_0 n^1 - n^1 \partial n^2)}{1 + n^3} + \frac{f_t^2}{2} (\partial_0 n^a)^2 - \frac{f^2}{2} (D_i n^a)^2 - \mu W^{ab} n_a n_b$$

EFT of chiral magnets

$$\hat{H} = \sum_n \sum_{i=1}^{d} \left[\frac{J}{2} (\hat{\boldsymbol{s}}^{n+\hat{i}} - \hat{\boldsymbol{s}}^n)^2 + \boldsymbol{D}_i \cdot (\hat{\boldsymbol{s}}^n \times \hat{\boldsymbol{s}}^{n+\hat{i}}) \right] + \sum_n (\hat{\boldsymbol{s}}^n)^t C \hat{\boldsymbol{s}}^n$$

Low-energy limit

$$\mathcal{L}_{\mathrm{eff}} = \frac{m(n^2 \partial_0 n^1 - n^1 \partial n^2)}{1+n^3} + \frac{f_t^2}{2} (\partial_0 n^a)^2 - \frac{f^2}{2} (D_i n^a)^2 - \mu W^{ab} n_a n_b$$

with

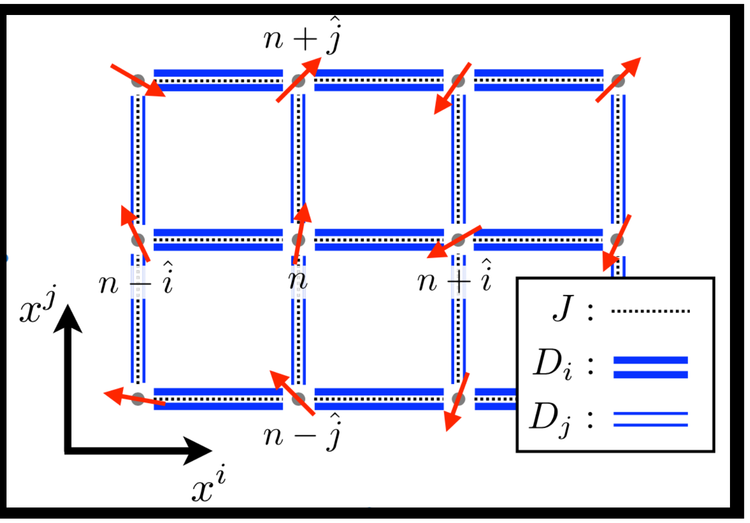
$$\begin{cases} D_i n^a \equiv \partial_i n^a - \epsilon^a{}_{bc} n^b A_i^c \\ A_i^a = (Ja)^{-1} D_i^a, \quad W = C - C_{\mathrm{cr}} \quad \text{with} \quad C_{\mathrm{cr}} \equiv \frac{1}{2J} (D_i^a t_a)^2 \end{cases}$$

- $m \neq 0, f_t^2 = 0$: Ferromagnets with magnetization vector n^a
- $m = 0, f_t^2 \neq 0$: Anti-ferromagnets with Néel vector n^a
- $m \neq 0, f_t^2 \neq 0$: Ferrimagnets with Néel vector n^a

Outline

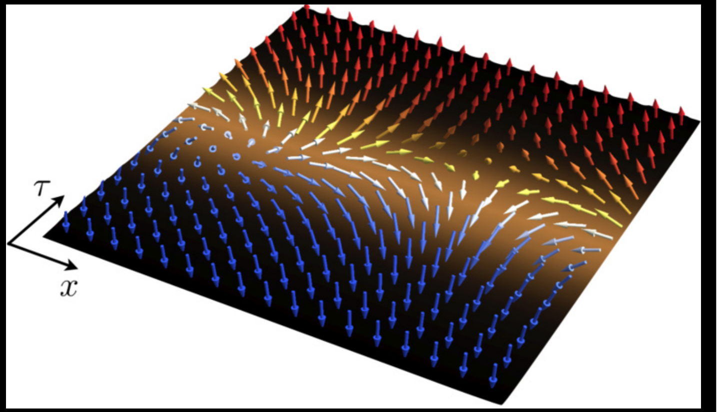
Formulation:

Background field (spurion) method
for O(3) nonlinear sigma model



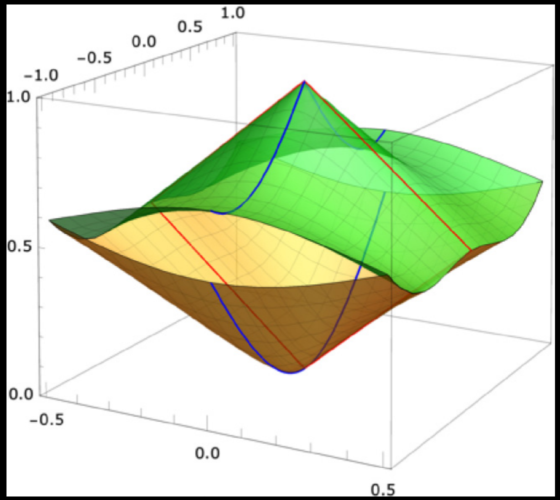
Instantons in 1+1d antiferro magnet :

Various instanton solutions
Equivalence theorem



Helical/spiral phases and NG modes:

Inhomogeneous ground states
Several types of NG modes



Antiferromagnetic spin chain

◆ **Effective Lagrangian** — Staggered mag. field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu n^a)^2 + \kappa(n^1\partial_x n^2 - n^2\partial_x n^1) + \frac{\mu}{2}[1-(n^3)^2] + Bn^3$$

One-dimensional DM int. + Anisotropic pot.

◆ **Classical ground state**

$$E[n] = \int \mathrm{d}x \left[\frac{(D_x n^a)^2}{2} + (\mu-\kappa^2)\frac{1-(n^3)^2}{2} + Bn^3 \right]$$ is minimized

if $D_x n^a = 0 \quad (a=1,2,3)$ and $(\mu-\kappa^2)\frac{1-(n^3)^2}{2} + Bn^3$ is minimized

$$\Rightarrow \begin{cases} n^1 + \mathrm{i}n^2 = A\mathrm{e}^{-\mathrm{i}\kappa x} \\ n^3 = \pm\sqrt{1-|A|^2} \end{cases}$$

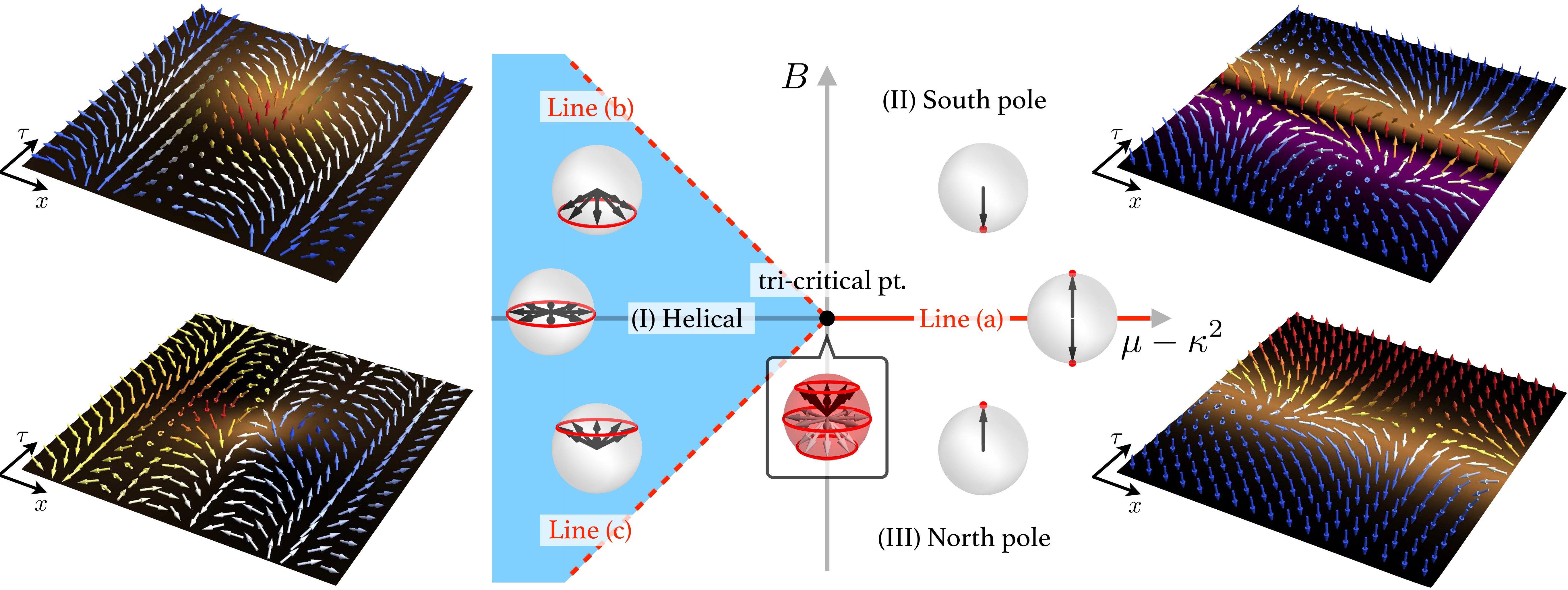
$$= \frac{\mu-\kappa^2}{2}|A|^2 \pm B\sqrt{1-|A|^2}$$

Once we fix the parameters $(\mu-\kappa^2, B)$

we can easily find the minimizer A!!

Phase diagram and instantons

◆ Phase diagram and order parameter manifolds



	OPM G/H	π_0	π_1	π_2	π_3
(I)	S^1	0	$\mathbb{Z}$	0	0
(a)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
Tricritical point	$O(3)/O(2) \simeq S^2$	0	0	$\mathbb{Z}$	$\mathbb{Z}$
(II) (III) (b)(c)	1 point	0	0	0	0

BPS solution → Tricritical point

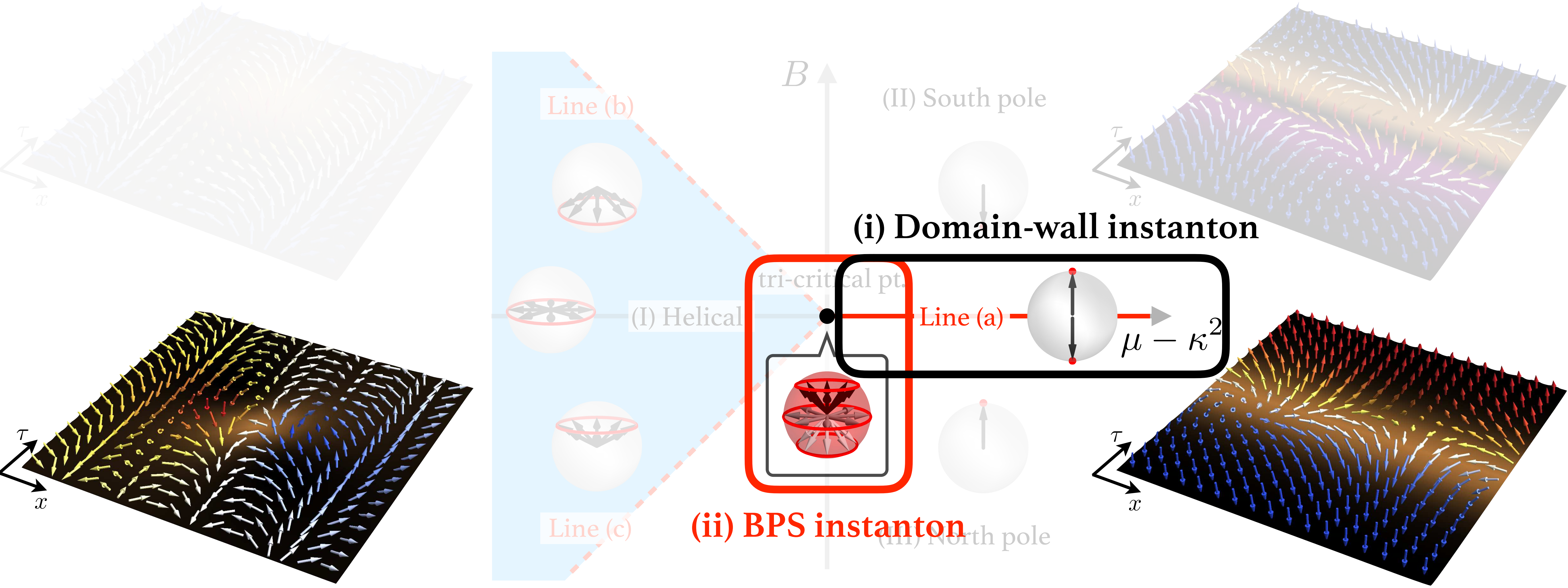
Vortex solution → π_1 of (I): $\mathbb{Z}$

Domain-wall solution → π_0 of (a): $\mathbb{Z}_2$

Bion solution → (II) (III) (b)(c)

Phase diagram and instantons

◆ Phase diagram and order parameter manifolds



BPS solution → Tricritical point

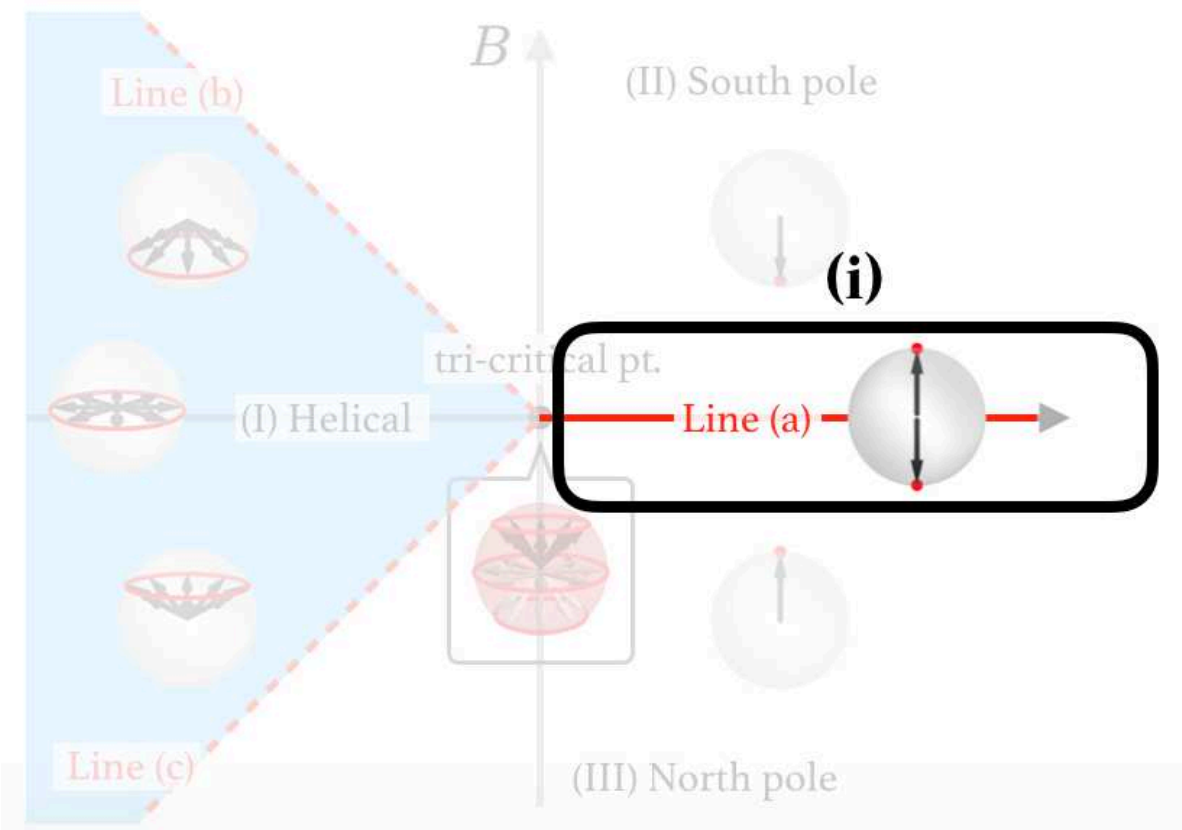
Bion solution → (II) (III) (b)(c)

	OPM G/H	π_0	π_1	π_2	π_3
(I)	S^1	0	$\mathbb{Z}$	0	0
(a)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
Tricritical point	$O(3)/O(2) \simeq S^2$	0	0	$\mathbb{Z}$	$\mathbb{Z}$
(II) (III) (b)(c)	1 point	0	0	0	0

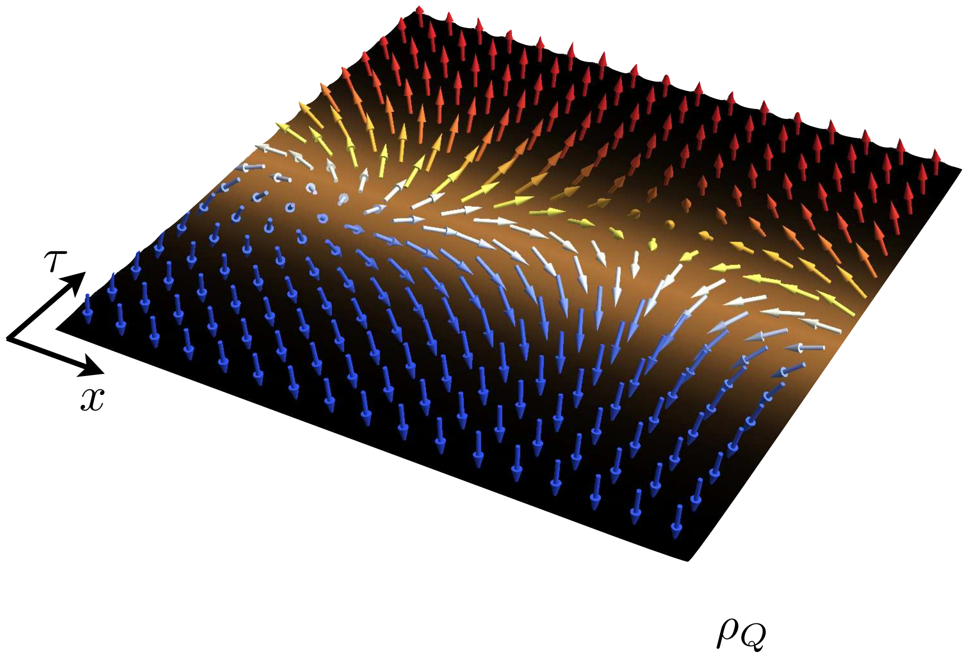
Vortex solution → π_1 of (I): $\mathbb{Z}$

Domain-wall solution → π_0 of (a): $\mathbb{Z}_2$

(i) Domain-wall instanton



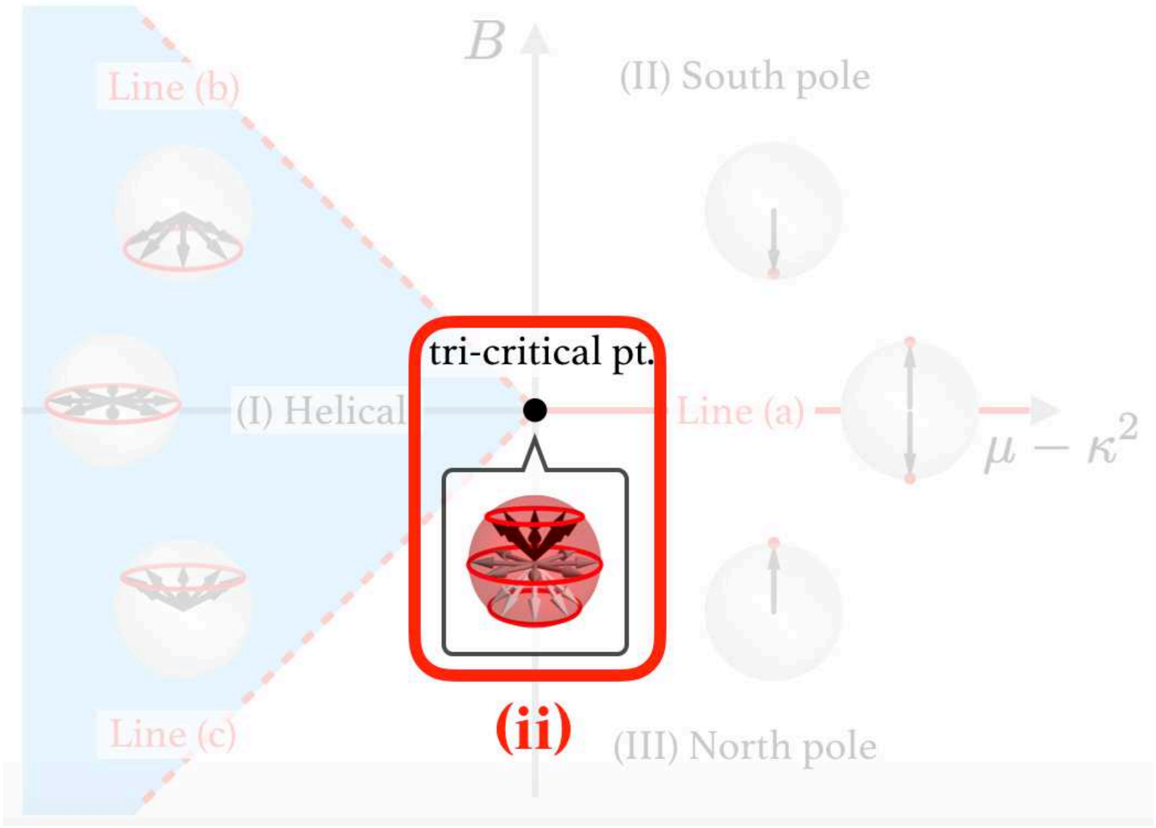
$$v \equiv \frac{n^1 + \mathrm{i}n^2}{1 + n^3} = C\mathrm{e}^{-\mathrm{i}\kappa x - \sqrt{\mu - \kappa^2}\tau}, \quad C \in \mathbb{C}$$



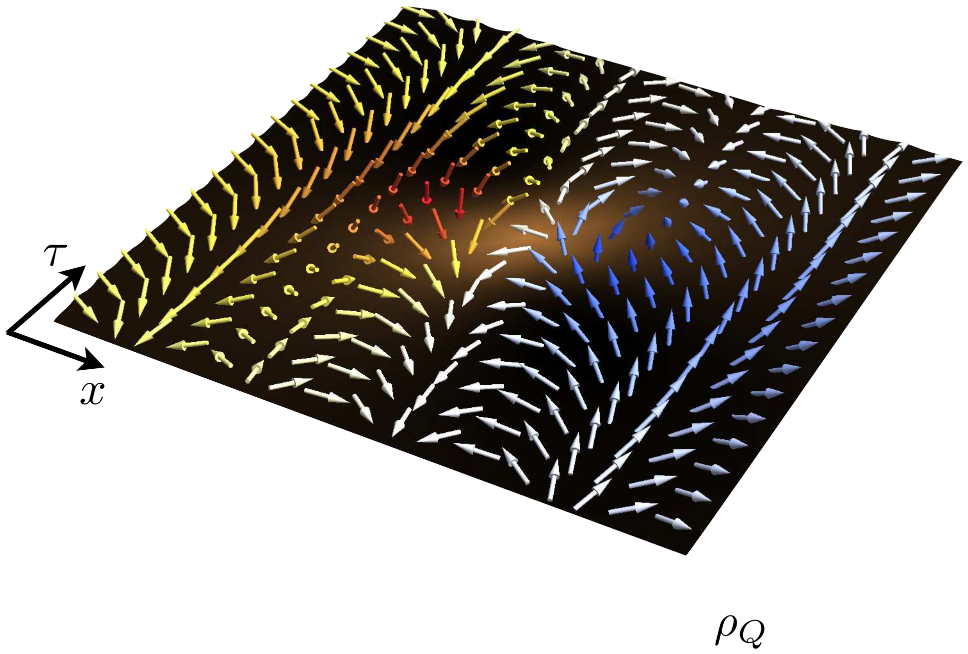
◆ **Properties**

- BPS solutions obtained by **Bogomorni completion!**
- **Helical configuration & finite instanton charge** arise on domain wall!!

(ii) **BPS solutions** at tricritical pt.

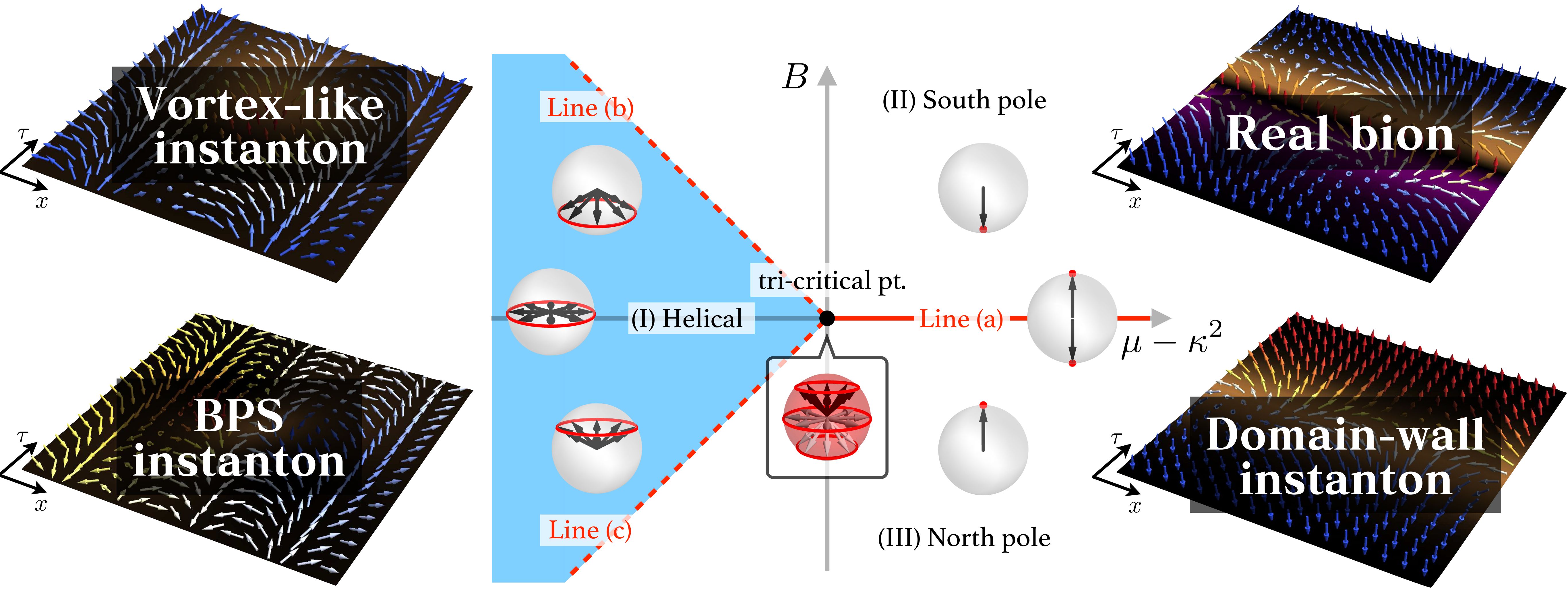


$$v_{\mathrm{inst}} = \mathrm{e}^{-\mathrm{i}\kappa x} \frac{p(z)}{q(z)} \quad \textbf{with} \quad z = x + \mathrm{i}\tau$$



Phase diagram and instantons

◆ Phase diagram and order parameter manifolds



BPS solution

Vortex solution

	OPM G/H	π_0	π_1	π_2	π_3
(I)	S^1	0	$\mathbb{Z}$	0	0
(a)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
Tricritical point	$O(3)/O(2) \simeq S^2$	0	0	$\mathbb{Z}$	$\mathbb{Z}$
(II) (III) (b)(c)	1 point	0	0	0	0

Bion solution

Domain-wall solution

Equivalence theorem

Q. Why we can exhaustively construct instanton solutions

A. Equivalence to the model without DM interaction!

◆ O(3) NLσ model with DM interaction —— Staggered mag. field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu n^a)^2 + \kappa(n^1\partial_x n^2 - n^2\partial_x n^1) + \frac{\mu}{2}[1-(n^3)^2] + Bn^3$$

One-dimensional DM int. + Anisotropic pot.

One-to-one correspondence with

$$n^1 + \mathrm{i}n^2 = (\hat{n}^1 + \mathrm{i}\hat{n}^2)\mathrm{e}^{-\mathrm{i}\kappa x}, \quad n^3 = \hat{n}^3$$

◆ O(3) NLσ model with DM interaction —— Staggered mag. field

$$\mathcal{L}_{\mathrm{woDM}} = \frac{1}{2}(\partial_\mu \hat{n}^a)^2 + \frac{\mu-\kappa^2}{2}[1-(\hat{n}^3)^2] + B\hat{n}^3$$

Anisotropic potential

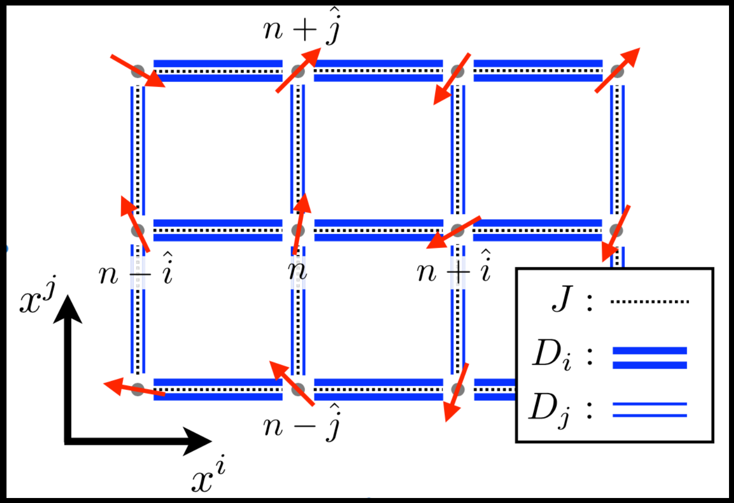
[cf. Kaplan-Shekhtman-Aharony-Entin-Wohlman (KSAE) int.]

Outline

Formulation:

Background field (spurion) method

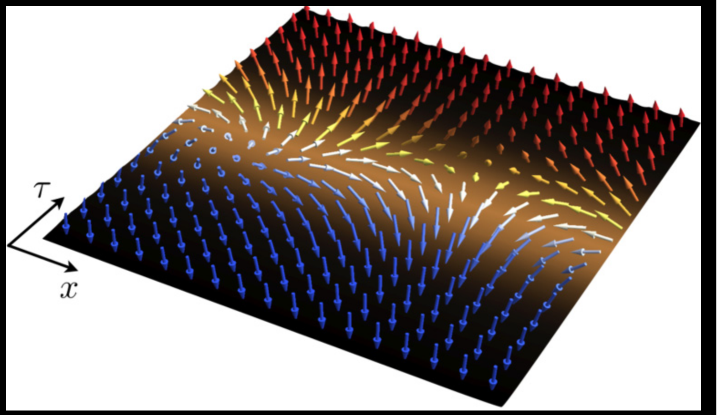
for O(3) nonlinear sigma model



Instantons in 1+1d antiferro magnet :

Various instanton solutions

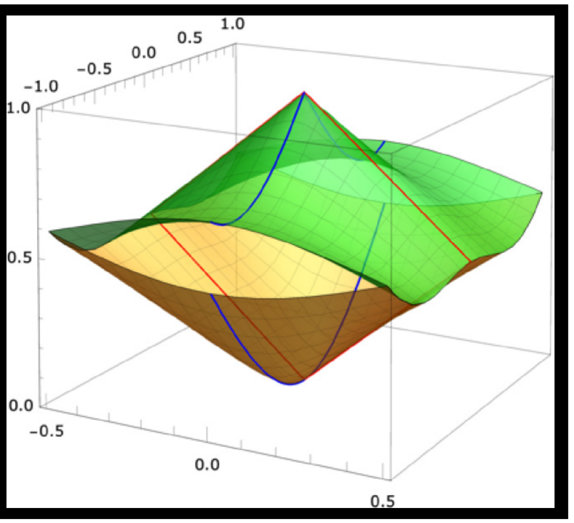
Equivalence theorem



Helical/spiral phases and NG modes:

Inhomogeneous ground states

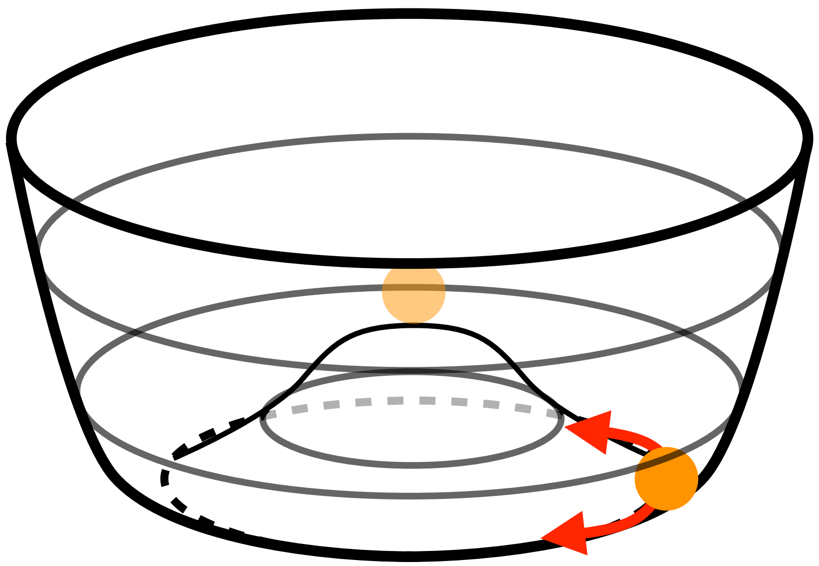
Several types of NG modes

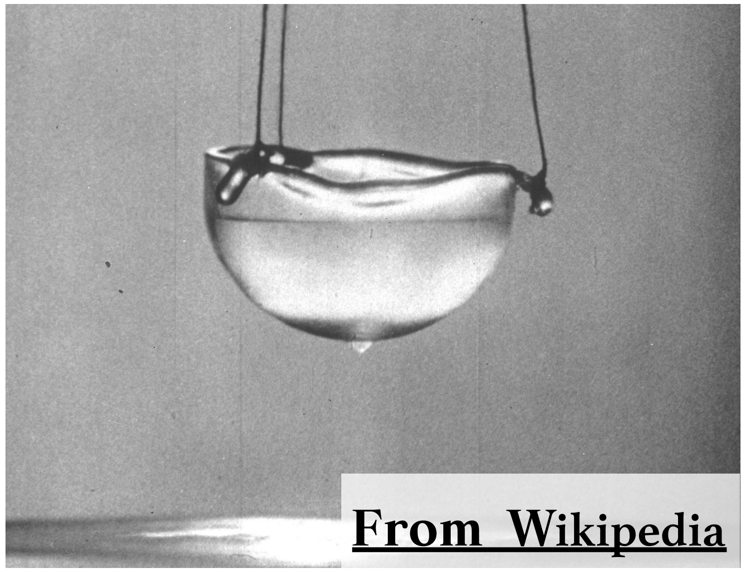


Primer to Nambu-Goldstone mode

SSB and NG mode

◆ U(1) symmetry breaking : superfluid phonon



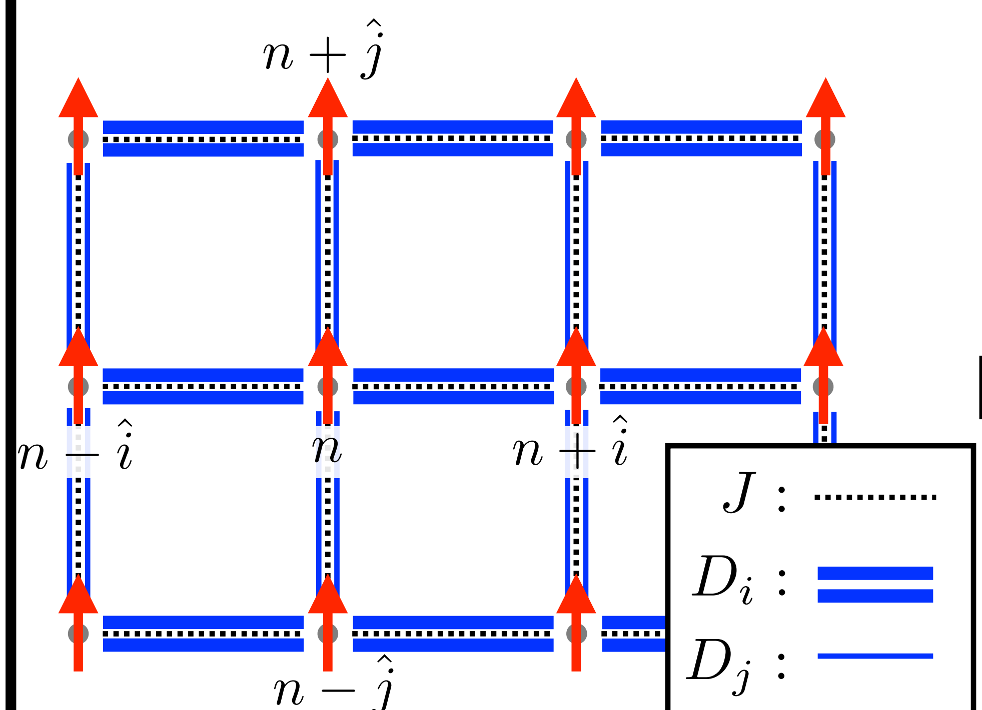


Energy spectrum & # of NG modes

$$\omega = ck$$

Same as number of broken symmetries

◆ O(3) symmetry breaking : ferromagnon



Energy spectrum & # of NG modes

$$\omega = ak^2$$

Different from number of broken symmatries

Classification of NG modes

Continuous symmetry $\Leftrightarrow$ **Charge** $\hat{Q}_a$

◆ **Definition of SSB**

$$\exists\,\Phi_i(x) \ \textbf{satisfying} \ \langle\delta_a\Phi_i(x)\rangle = \langle[\mathrm{i}\hat{Q}_a, \hat{\Phi}_i(x)]\rangle \neq 0$$

Classification of NG modes

SSB of continuous symmetry

⬇

There is a gapless NG mode

[Nambu,Goldstone, 1960-1962]

Classification of NG modes

Continuous symmetry $\Leftrightarrow$ Charge $\hat{Q}_a$

◆ Definition of SSB

$$\exists\, \Phi_i(x) \text{ satisfying } \langle \delta_a \Phi_i(x) \rangle = \langle [\mathrm{i}\hat{Q}_a, \hat{\Phi}_i(x)] \rangle \neq 0$$

◆ Classification of NG modes

[Hidaka (2012), Watanabe-Murayama(2012)]

- Type-A NG mode : $\forall \hat{Q}_b$ について $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle = 0$

  # of broken symmetries = # of NG modes with $\omega = ck$

- Type-B NG mode : $\exists\, \hat{Q}_b$ such that $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle \neq 0$

  # of broken symmetries ≠ # of NG modes with $\omega = ak^2$

Nonrelativistic NG mode

Superfluid phonon

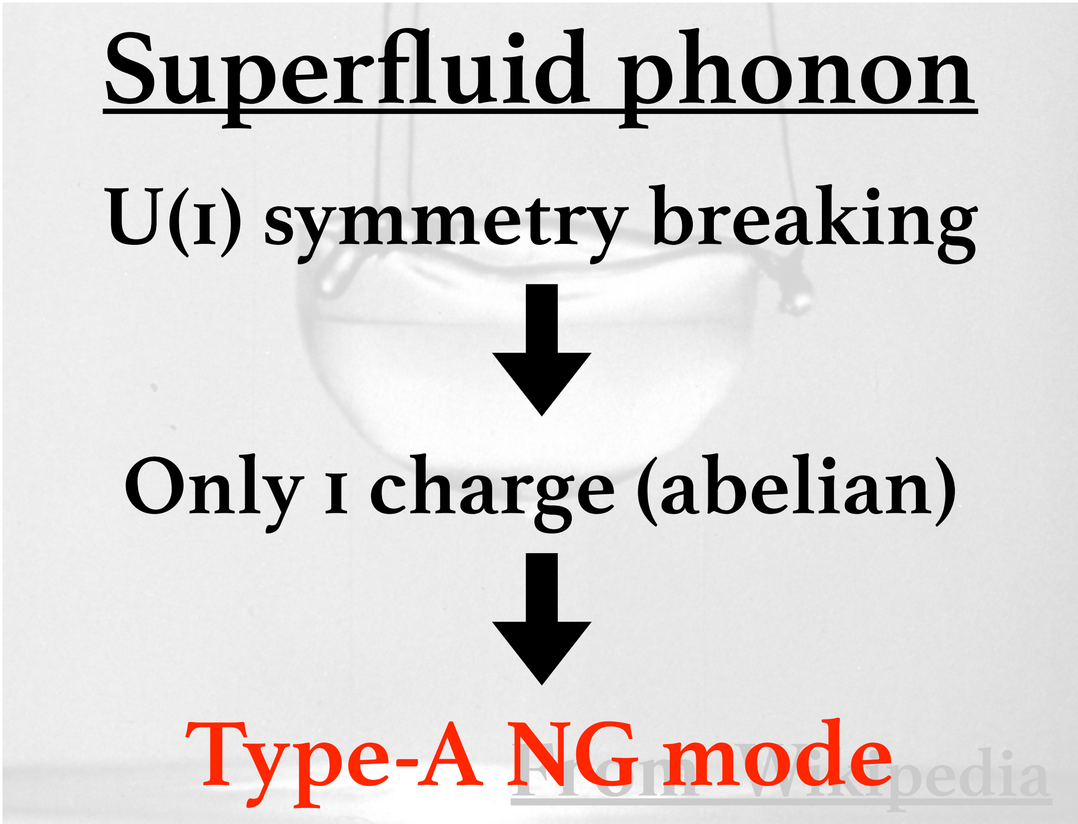
U(1) symmetry breaking

↓

Only 1 charge (abelian)

↓

Type-A NG mode



Ferromagnon

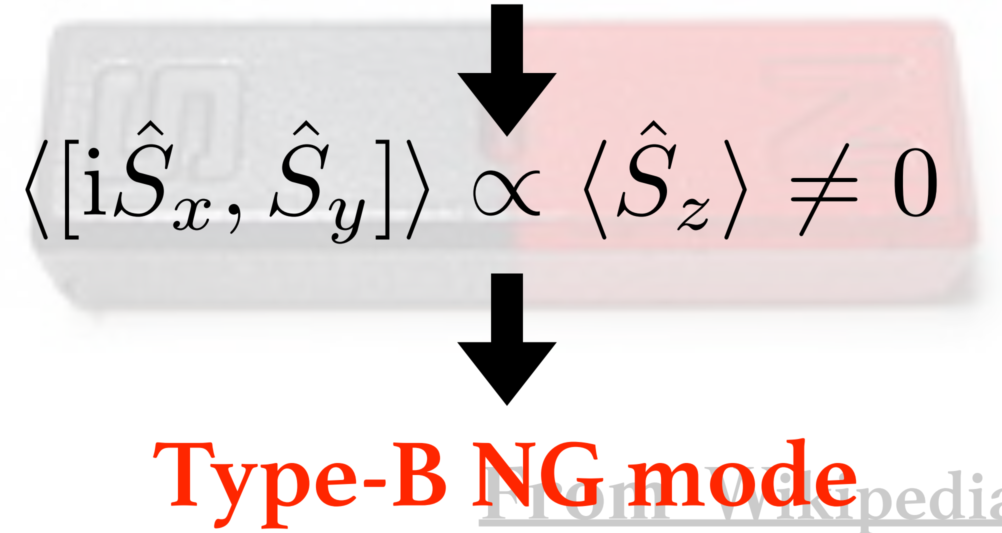
O(3) symmetry breaking

↓

$$\langle [\mathrm{i}\hat{S}_x, \hat{S}_y] \rangle \propto \langle \hat{S}_z \rangle \neq 0$$

↓

Type-B NG mode



◆ Classification of NG modes

[Hidaka (2012), Watanabe-Murayama(2012)]

- Type-A NG mode : $\forall \hat{Q}_b$ について $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle = 0$

  \# of broken symmetries = # of NG modes with $\omega = ck$

- Type-B NG mode : $\exists \hat{Q}_b$ such that $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle \neq 0$

  \# of broken symmetries ≠ # of NG modes with $\omega = ak^2$

Nonrelativistic NG mode

Superfluid phonon

U(1) symmetry breaking

↓

Only 1 charge (abelian)

↓

Type-A NG mode

From Wikipedia

Ferromagnon

O(3) symmetry breaking

↓

$$\langle [i\hat{S}_x, \hat{S}_y] \rangle \propto \langle \hat{S}_z \rangle \neq 0$$

↓

Type-B NG mode

From Wikipedia

◆ Effective Lagrangian

[Leutwyler (1994), Watanabe-Murayama(2012)]

$$\mathcal{L}_{\rm eff} = m^\alpha f_{\alpha ab} \pi^a \partial_0 \pi^b + \frac{f^2}{2} g_{ab} \partial_0 \pi^a \partial_0 \pi^b - \frac{1}{2} g_{ab} \partial^i \pi^a \partial_i \pi^b + \cdots$$

$\simeq \langle [i\hat{Q}_a, \hat{Q}_b] \rangle$: Term peculiar to nonrelativistic system

➡ **Unified description including Type-B NG mode**

Open problem

◆ Classification of NG modes

[Hidaka (2012), Watanabe-Murayama(2012)]

- Type-A NG mode : $\forall \hat{Q}_b$ について $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle = 0$

  # of broken symmetries = # of NG modes with $\omega = ck$

- Type-B NG mode : $\exists \hat{Q}_b$ such that $\langle [\mathrm{i}\hat{Q}_a, \hat{Q}_b] \rangle \neq 0$

  # of broken symmetries ≠ # of NG modes with $\omega = ak^2$

This is a general theorem for internal (on-site) symmetry!

What happens for NG modes associated with spontaneous spacetime symmetry breaking?

⇒ Let's investigate inhomogeneous phases of chiral magnets!

Inhomogeneous phase 1: Helical phase

[uniaxial DM int. +anisotropy]

Helical phase

◆ **Effective Lagrangian for uniaxial anisotropic chiral magnet**

$$A_i^a = \kappa_i \delta_3^a, \ \ \ell W^{ab} = \frac{W}{2}\delta_3^a \delta_3^b$$ **: Uniaxial anisotropy**

Anisotropic pot.

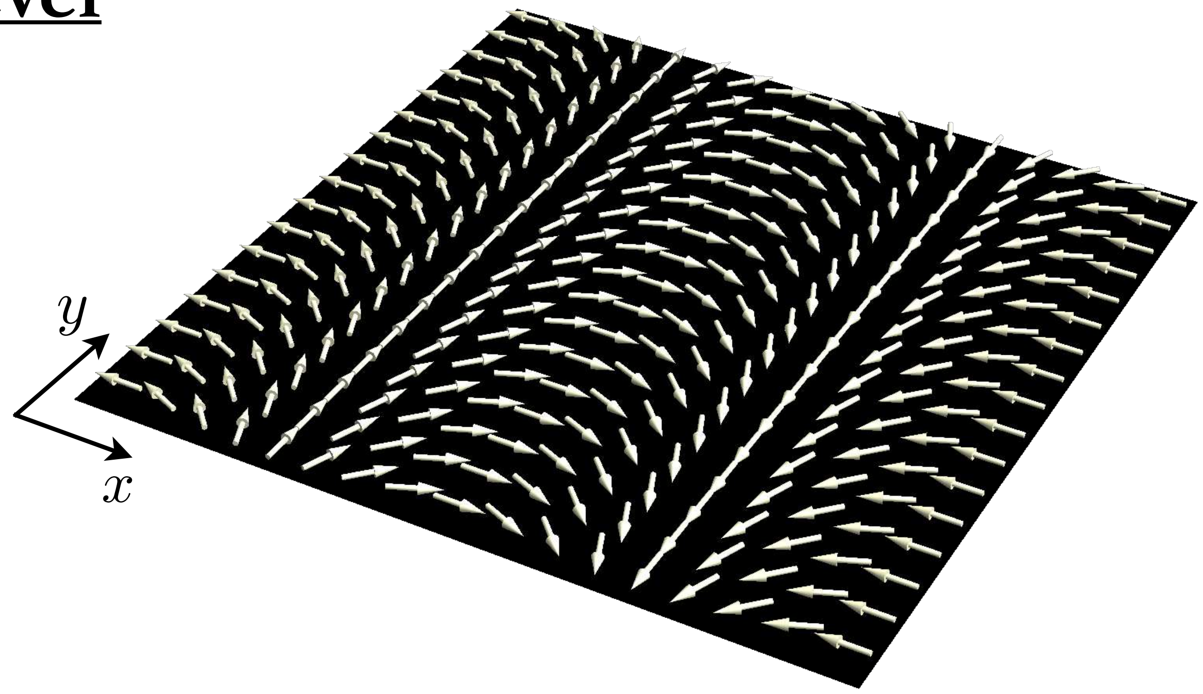
$$\mathcal{L}_{\rm eff} = \frac{m(n^2\partial_0 n^1 - n^1 \partial_0 n^2)}{1+n^3} + \frac{f_t^2}{2}(\partial_0 n^a)^2 - \frac{f_s}{2}(\partial_i n^a - \kappa_i \epsilon^a{}_{b3} n^b)^2 + \frac{W}{2}(n^3)^2$$

DM interaction

Inhomogeneous ground state at tree-level

$$\bar{n}^a = \begin{pmatrix} \sqrt{1-\bar{A}^2}\cos(\boldsymbol{\kappa}\cdot\boldsymbol{x} + \bar{\phi}) \\ -\sqrt{1-\bar{A}^2}\sin(\boldsymbol{\kappa}\cdot\boldsymbol{x} + \bar{\phi}) \\ \bar{A} \end{pmatrix}$$

with $$\bar{A} = \begin{cases} \pm 1 & \text{for} \quad W > 0, \\ 0 & \text{for} \quad W < 0, \\ \text{arbitrary} \in [-1,1] & \text{for} \quad W = 0. \end{cases}$$



Inhomogeneous phase at $W < 0$

NG mode in helical phase

◆ **Effective Lagrangian for fluctuation field {δϕ, δA} in helical phase**

$$\mathcal{L}_{\mathrm{eff}}^{(2)} = m(1-\delta A)\partial_0\delta\phi + \frac{f_{\mathrm{t}}^2}{2}[(\partial_0\delta A)^2 + (\partial_0\delta\phi)^2] - \frac{f_{\mathrm{s}}^2}{2}[(\partial_i\delta A)^2 + (\partial_i\delta\phi)^2] + \frac{W}{2}(\delta A)^2,$$

◆ **Dispersion relation**

- **Antiferromagnet** ($f_t \neq 0, m=0$) : $\omega = \frac{f_{\mathrm{s}}}{f_{\mathrm{t}}}|\boldsymbol{k}|, \quad \frac{\sqrt{|W| + (f_{\mathrm{s}}\boldsymbol{k})^2}}{f_{\mathrm{t}}},$

- **Ferromagnet** ($f_t = 0, m \neq 0$) : $\omega = \frac{f_{\mathrm{s}}|\boldsymbol{k}|\sqrt{|W| + (f_{\mathrm{s}}\boldsymbol{k})^2}}{m},$

- **Ferrimagnet** ($f_t \neq 0, m=0$) : $\omega = \begin{cases} \left(\frac{|W|}{m^2+|W|}\right)^{\frac{1}{2}} \frac{f_{\mathrm{s}}}{f_{\mathrm{t}}}|\boldsymbol{k}| + \frac{m^4}{2\sqrt{|W|(m^2+|W|)^5}} \frac{(f_{\mathrm{s}}|\boldsymbol{k}|)^3}{f_{\mathrm{t}}} + O(|\boldsymbol{k}|^5), \\ \frac{\sqrt{m^2+|W|}}{f_{\mathrm{t}}} + \frac{2m^2+|W|}{2(m^2+|W|)^{3/2}} \frac{f_{\mathrm{s}}^2\boldsymbol{k}^2}{f_{\mathrm{t}}} + O(\boldsymbol{k}^4). \end{cases}$

All magnets show a linear isotropic dispersion relation $\omega \propto |\boldsymbol{k}|$

(interpreted as a translational phonon or magnon in a rotating frame)

Inhomogeneous phase 2: Spiral phase

[isotropic DM interaction]

Spiral phase

◆ **Effective Lagrangian for chiral magnet with isotropic DM int.**

$A_i^a = -\kappa\delta_i^a, \ \ell W^{ab} n^a n^b = -f_s^2\kappa^2 (n^3)^2$ **: Isotropic DM interaction**

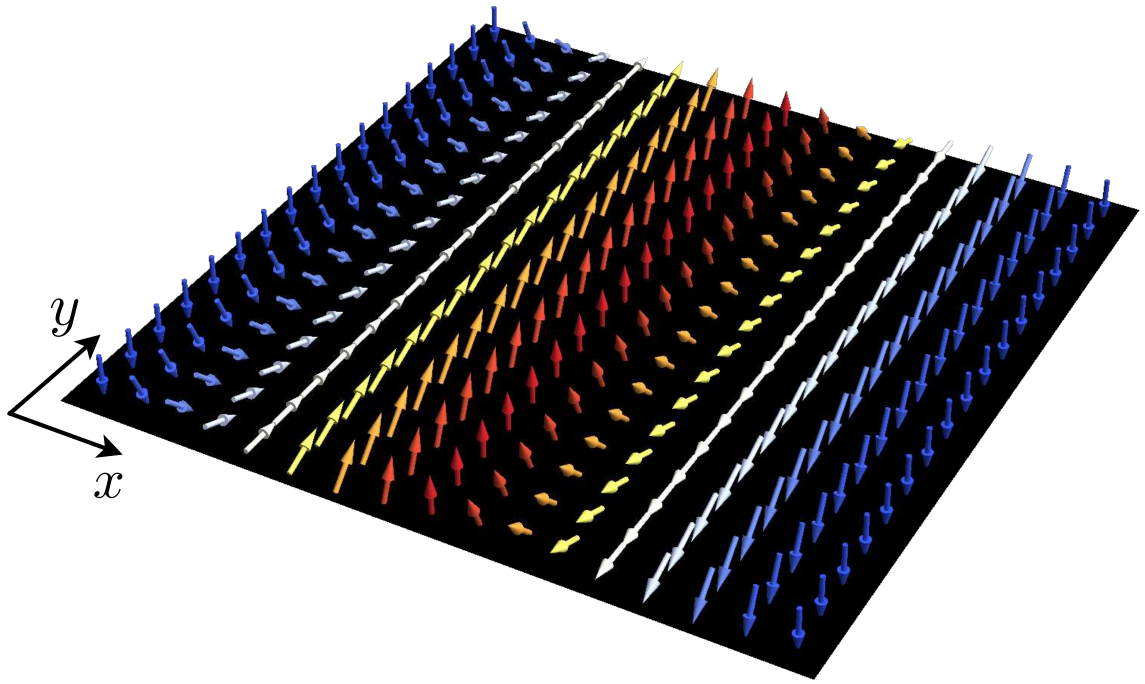
$$\mathcal{L}_{\rm eff} = \frac{m(n^2\partial_0 n^1 - n^1\partial_0 n^2)}{1+n^3} + \frac{f_t^2}{2}(\partial_0 n^a)^2 - \frac{f_s}{2}(\partial_i n^a)^2 + f_s^2\kappa^2\left[n^3(\partial_y n^1 - \partial_x n^2) + (n^2\partial_x - n^1\partial_y)n^3\right]$$

DM interaction

Inhomogeneouos ground state at tree-

Energy is minimized by

$$\bar{n}^a = \begin{pmatrix} 0 \\ \sin(-\kappa x + \bar{\theta}) \\ \cos(-\kappa x + \bar{\theta}) \end{pmatrix}$$



[Note. DM int. does not contributes to eom, but does to energy min. condition]

Energy spectrum in spiral phase

◆ Effective Lagrangian for fluctuation field {δϕ, δA} in spiral phase

$$\mathcal{L}_{\rm eff}^{(2)} = \frac{f_{\rm t}^2}{2}\left[(\partial_0\delta\theta)^2 + (\partial_0\delta\Omega)^2\right] - \frac{f_{\rm s}^2}{2}\left[(\partial_i\delta\theta)^2 + (\partial_i\delta\Omega)^2\right] - \frac{f_{\rm s}^2\kappa^2}{2}(\delta\Omega)^2 + m\delta\theta\partial_0\delta\Omega - 2f_{\rm s}^2\kappa\sin\kappa x\,\delta\theta\partial_y\delta\Omega.$$

←*x*-dependent term

◆ Equation of motion

$$\begin{pmatrix} f_t^2\omega^2 & -im\omega \\ im\omega & f_t^2\omega^2 \end{pmatrix}\begin{pmatrix}\delta\theta \\ \delta\Omega\end{pmatrix} = \begin{pmatrix} -\nabla^2 & 2\kappa\sin\kappa x\partial_y \\ -2\kappa\sin\kappa x\partial_y & -\nabla^2+\kappa^2 \end{pmatrix}\begin{pmatrix}\delta\theta \\ \delta\Omega\end{pmatrix}$$

To obtain the dispersion relation, the eigenvalue of this matrix↑ is needed!

➡ Equivalent to QM under the periodic pot. (with initernal d.o.f.)!!

Recalling BlochThm., we can solve this in the same way as the Band theory!

Truncated band approximation

Eigenvalue equation: $H(x)\vec{\varphi}_{k_x}(x) = E_{k_x}\vec{\varphi}_{k_x}(x)$ with $H(x + 2\pi/\kappa) = H(x)$

Expand the eigenvector as $\vec{\varphi}_{k_x}(x) = \int \frac{\mathrm{d}k_\perp}{2\pi} \sum_n \mathrm{e}^{\mathrm{i}(k_x+\kappa n)x + \mathrm{i}k_\perp y} \vec{v}_n(\boldsymbol{k})$

◆ **Recurrence relations**

$$f_\mathrm{s}^2 \left([(k_x + \kappa n)^2 + k_\perp^2] v_n^{(0)}(\boldsymbol{k}) + \kappa k_\perp [v_{n-1}^{(1)}(\boldsymbol{k}) - v_{n+1}^{(1)}(\boldsymbol{k})] \right) = E_n(\boldsymbol{k}) v_n^{(0)}(\boldsymbol{k})$$

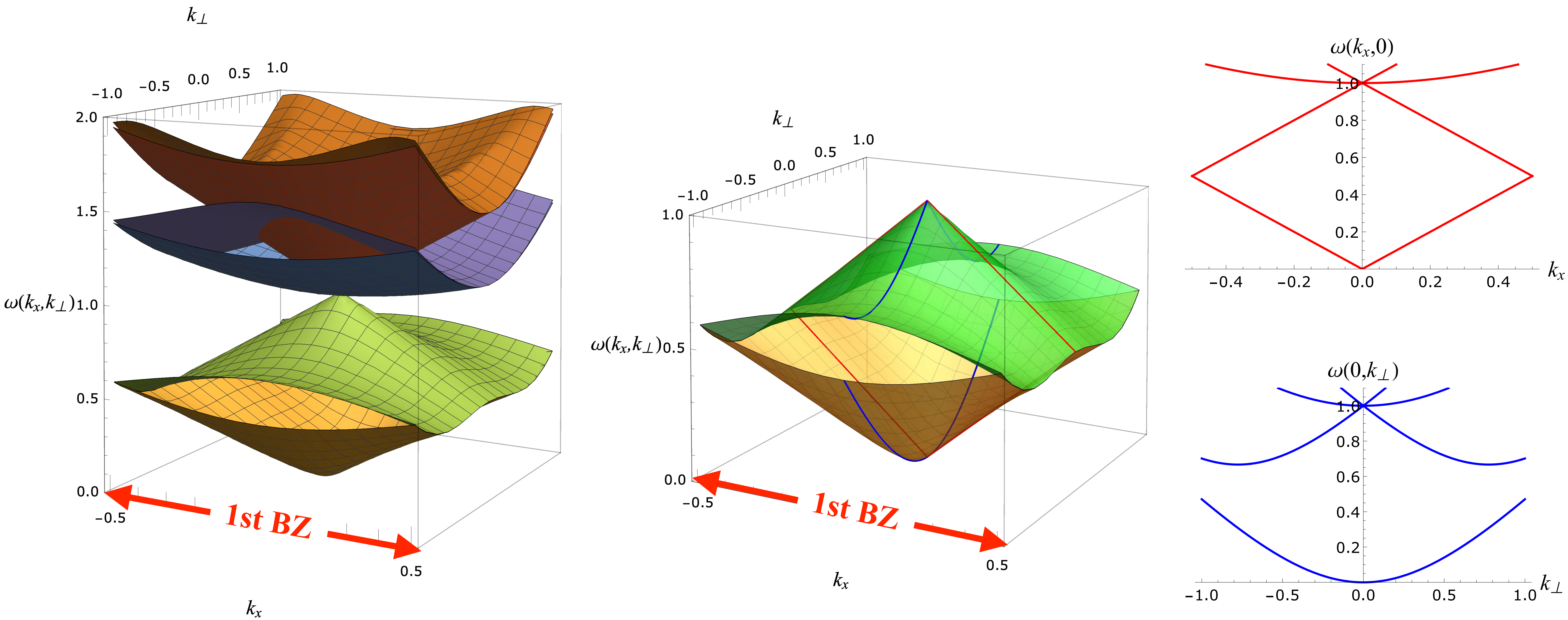
$$f_\mathrm{s}^2 \left(-\kappa k_\perp [v_{n-1}^{(0)}(\boldsymbol{k}) - v_{n+1}^{(0)}(\boldsymbol{k})] + [(k_x + \kappa n)^2 + k_\perp^2 + \kappa^2] v_n^{(1)}(\boldsymbol{k}) \right) = E_n(\boldsymbol{k}) v_n^{(1)}(\boldsymbol{k})$$

By truncating the band index, we can solve the eigenvalue problem!

For example, only by considering the three band, we can reduce the problem as

$$\begin{pmatrix} \omega_1^+ - \frac{E(\boldsymbol{k})}{f_\mathrm{s}^2} & -\kappa k_\perp & 0 \\ -\kappa k_\perp & \omega_0^- - \frac{E(\boldsymbol{k})}{f_\mathrm{s}^2} & \kappa k_\perp \\ 0 & \kappa k_\perp & \omega_{-1}^+ - \frac{E(\boldsymbol{k})}{f_\mathrm{s}^2} \end{pmatrix} \begin{pmatrix} v_1^{(1)}(\boldsymbol{k}) \\ v_0^{(0)}(\boldsymbol{k}) \\ v_{-1}^{(1)}(\boldsymbol{k}) \end{pmatrix} = 0, \quad \omega_n^+ \equiv (k_x + n\kappa)^2 + k_\perp^2 + \kappa^2$$

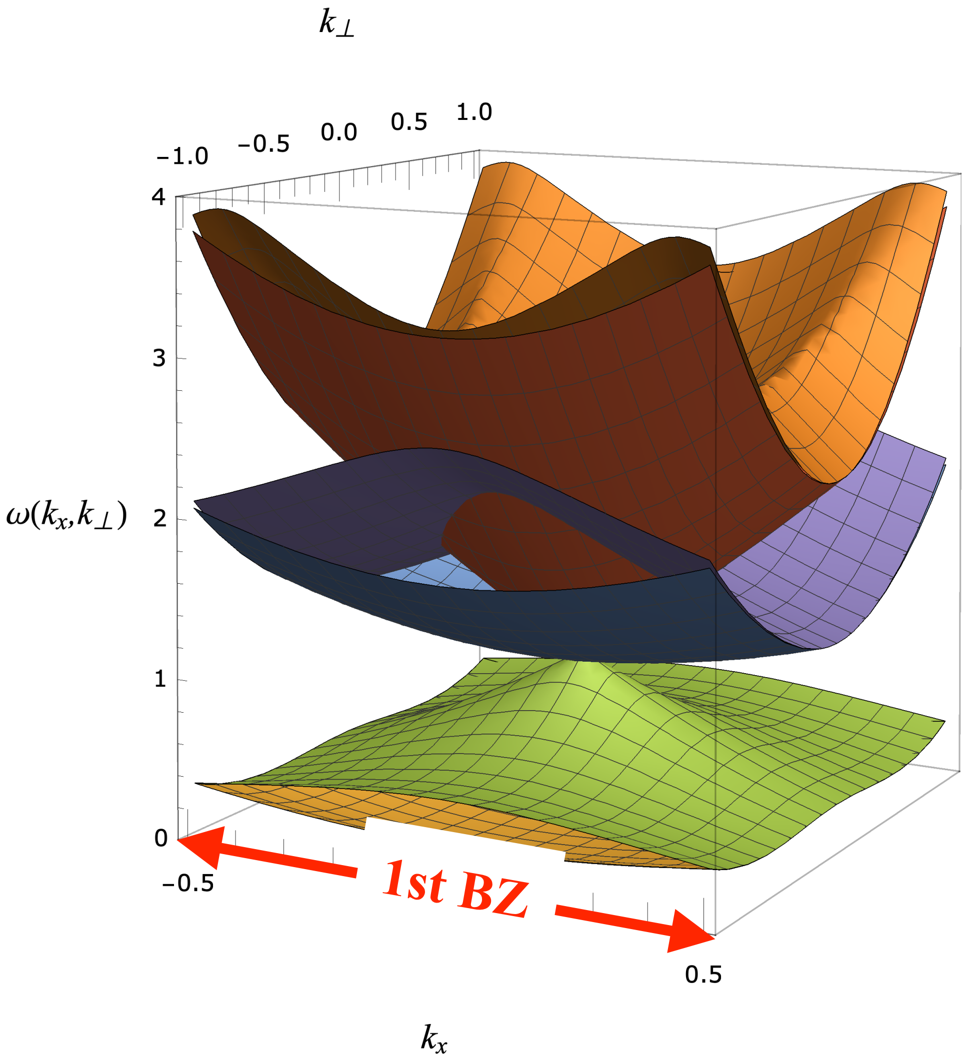
Antiferromagnetic spiral phase

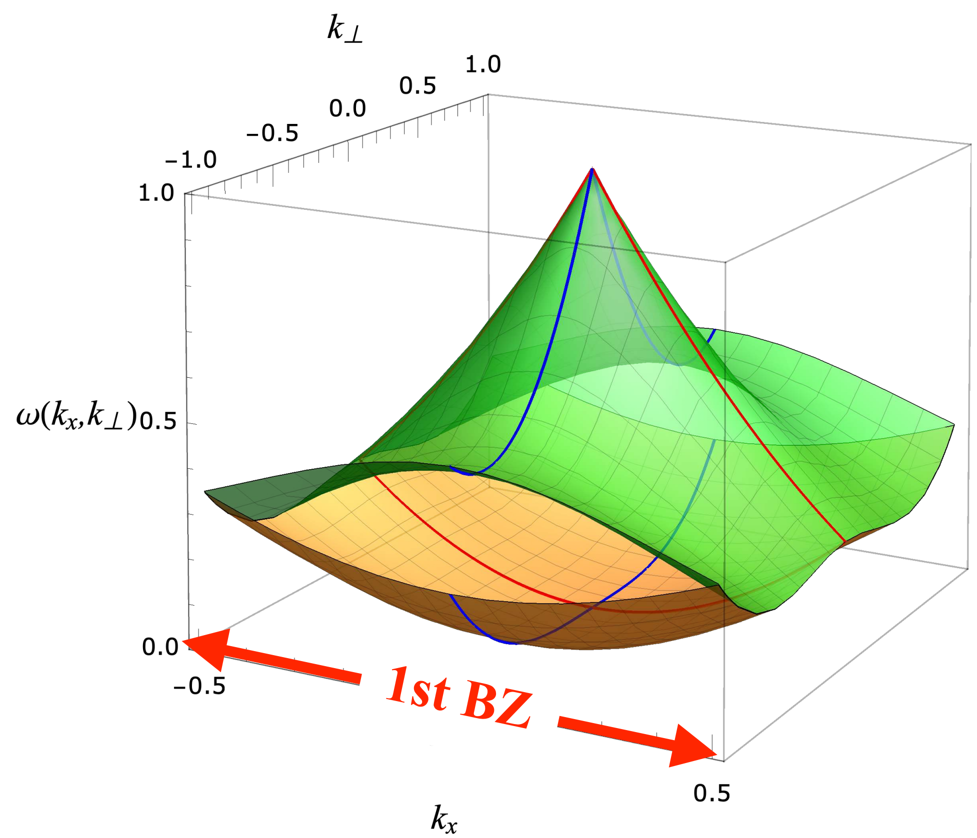


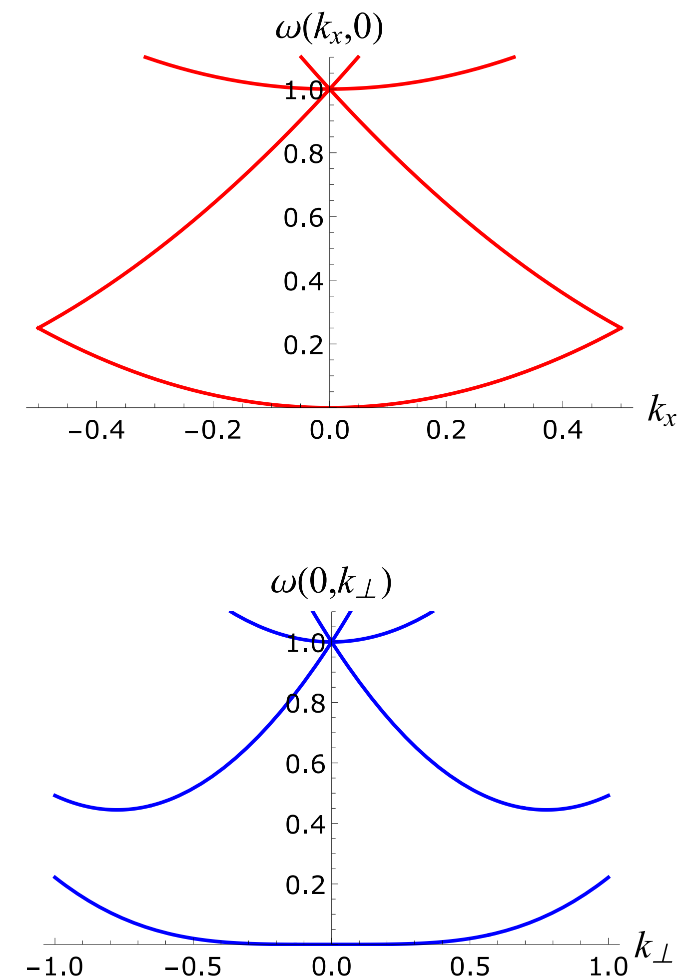
◆ Low-energy spectrum with anisotropy

$$\omega_{n=0}(\boldsymbol{k}) = \begin{cases} c_s|k_x|\left(1 - \dfrac{k_\perp^2}{2\kappa^2} + \dfrac{3k_\perp^4}{16\kappa^2|k_x|^2} + \cdots\right) & \text{if } k_x \neq 0, \\ c_s\sqrt{\dfrac{3}{8}\dfrac{k_\perp^2}{\kappa}} + \cdots & \text{if } k_x = 0, \end{cases}$$

Ferromagnetic spiral phase





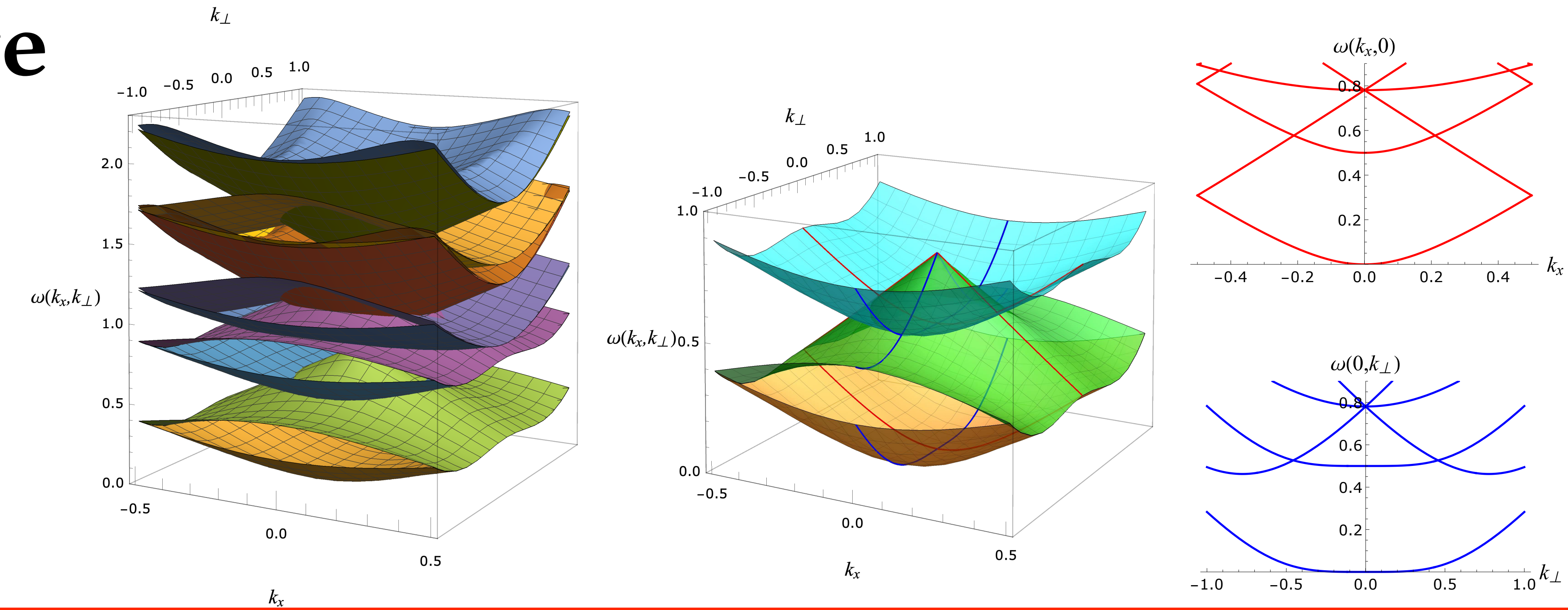


◆ Low-energy spectrum with anisotropy

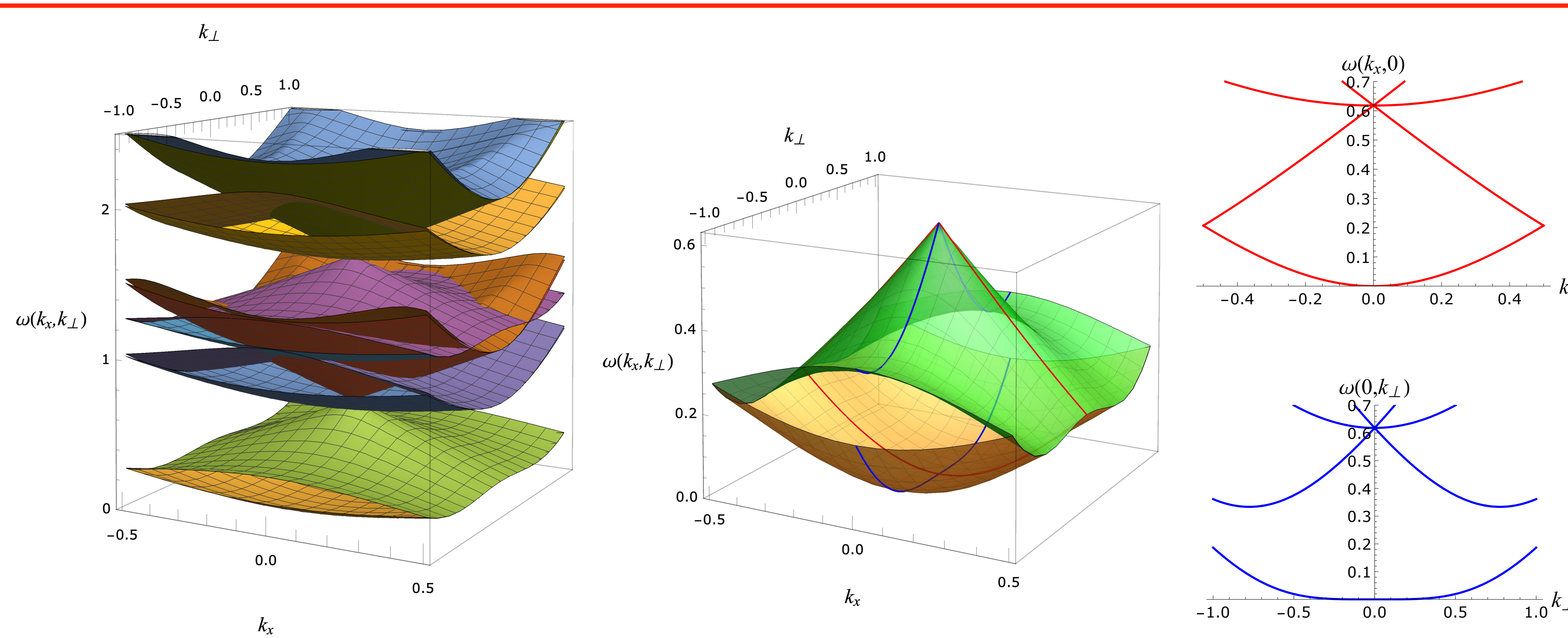
$$\omega_{n=0}(\boldsymbol{k}) = \frac{f_{\mathrm{s}}^2}{m}\left[k_x^2\left(1-\frac{k_\perp^2}{\kappa^2}\right)+\frac{k_\perp^4}{2\kappa^2}+\cdots\right].$$

Ferrimagnetic spiral phase

Large *m*



If $m\neq0$, the lowest band at small k is similar to the ferromag. case



Small *m*

Symmetry-based understanding

◆ NG mode in helical phase

All magnets show a linear isotropic dispersion relation $\omega \propto |\boldsymbol{k}|$

Symmetry breaking pattern: $\mathrm{SO}(2)_z \times \mathbb{R}^d \to \mathbb{R}_{s+\|} \times \mathbb{R}_\perp^{d-1}$

(NG mode = translational phonon or magnon in a rotating frame)

◆ NG mode in spiral phase

Anisotropic dispersion relation + dependence on type of magnets

Symmetry breaking pattern: $\mathrm{SO}(2)_{s+l} \times \mathbb{R}^2 \to \mathbb{R}_\perp^1$

Commutator btw charges: $\langle [\mathrm{i}P_x, \rho(x)] \rangle_{\mathrm{gs}} = -m\kappa \sin(-\kappa x + d)$

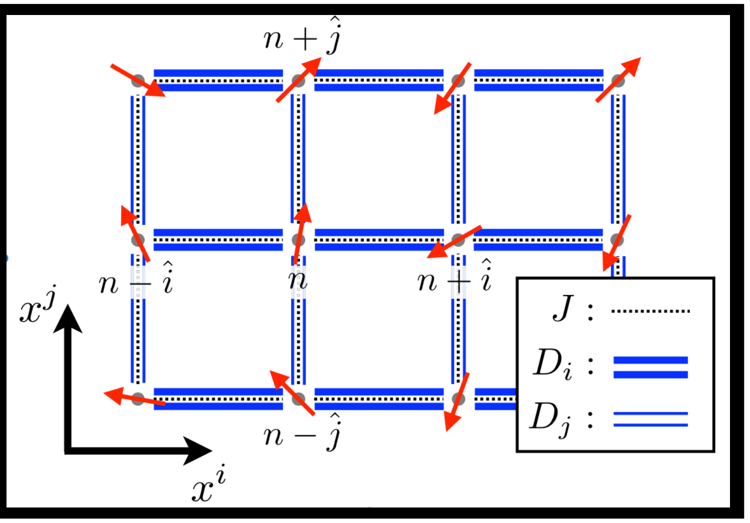
(NG modes = translational phonon and magnon)

Summary

Formulation:

Background field (spurion) method

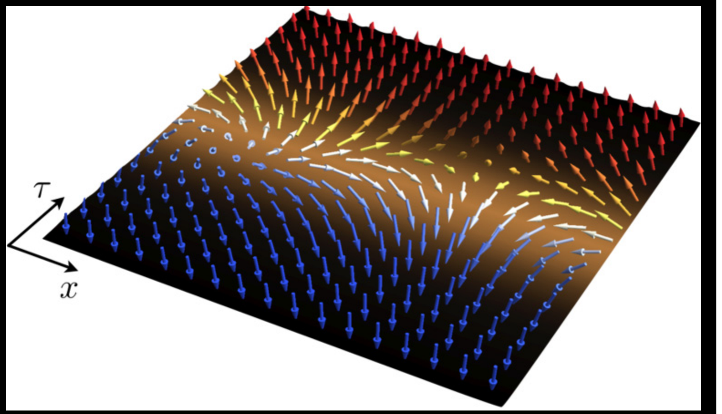
for O(3) nonlinear sigma model



Instantons in 1+1d antiferro magnet :

Various instanton solutions

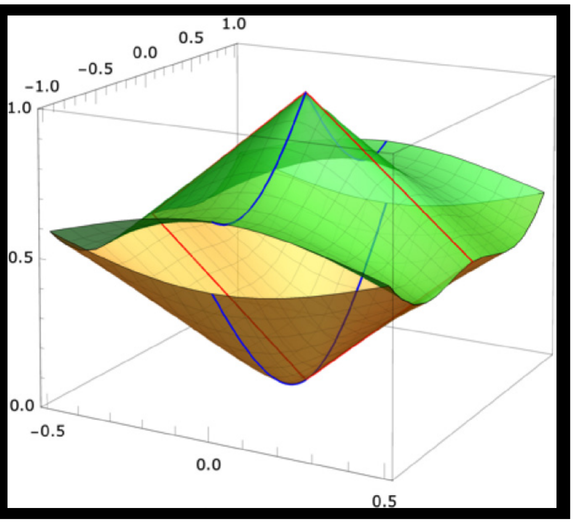
Equivalence theorem



Helical/spiral phases and NG modes:

Inhomogeneous ground states

Several types of NG modes



Outlook

Spins on Kagome-type lattice?

[Thermal Hall effect?]

Skyrmion current=electric current?

[Coupled dynamics with elemag?]

Skyrmion crystal?

[Multi-dim inhomogeneous phase]